

Mild Solutions and Harnack Inequality for Functional SPDEs with Dini Drift ^{*}

Xing Huang ^{a)}, Shao-Qin Zhang ^{b)}

a)School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China,

XingHuang@mail.bnu.edu.cn

b)School of Statistics and Mathematics, Central University of Finance and Economics, Beijing 100081, China,

zhangsq@cufe.edu.cn

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Abstract

The existence and uniqueness of the mild solution for a class of functional SPDEs with multiplicative noise and a locally Dini continuous drift are proved. In addition, under a reasonable condition the solution is non-explosive. Moreover, Harnack inequalities are derived for the associated semigroup under certain global conditions, which is new even in the case without delay.

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1 Introduction

Recently, using Zvonkin type transformation and gradient estimate, Wang [1] has proved the existence and uniqueness of the mild solution for a class of SPDEs with multiplicative noise and a locally Dini continuous drift. Following this, Wang and Huang [2] extend the results to a class of Functional SPDEs, where the drift without delay is assumed as Dini's continuity, and the delay drift is Lipschitzian under $\|\cdot\|_{\mathcal{C}_\nu}$, see the details in [2]. In this paper, we try to replace $L^2(\nu)$ norm in [2] with uniform norm (finite delay) or weighted uniform norm

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(infinite delay). Due to the technique reason, for example, the Fubini Theorem is unavailable in the present case, we need a stronger condition on the singular drift than that in [1], see **(a3)** in the following.

Let $(\mathbb{H}, \langle \cdot, \cdot \rangle, |\cdot|)$ and $(\bar{\mathbb{H}}, \langle \cdot, \cdot \rangle_{\bar{\mathbb{H}}}, |\cdot|_{\bar{\mathbb{H}}})$ be two separable Hilbert spaces. For any $r \in (0, \infty]$, let $\mathcal{C} = C((-\infty, 0] \cap [-r, 0]; \mathbb{H})$. For all $\xi \in \mathcal{C}$, define

$$\|\xi\|_{\infty} = \sup_{s \in (-\infty, 0] \cap [-r, 0]} (e^{-s} 1_{r=\infty} + 1_{r<\infty}) |\xi(s)|.$$

For any $f \in C((-\infty, \infty) \cap [-r, \infty); \mathbb{H})$, $t \geq 0$, let $f_t(s) = f(t+s)$, $s \in (-\infty, 0] \cap [-r, 0]$, then $f_t \in \mathcal{C}$.

Let $W = (W(t))_{t \geq 0}$ be a cylindrical Brownian motion on $\bar{\mathbb{H}}$ with respect to a complete filtration probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. More precisely, $W(\cdot) = \sum_{n=1}^{\infty} \bar{W}^n(\cdot) \bar{e}_n$ for a sequence of independent one dimensional Brownian motions $\{\bar{W}^n(\cdot)\}_{n \geq 1}$ with respect to $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, where $\{\bar{e}_n\}_{n \geq 1}$ is an orthonormal basis on $\bar{\mathbb{H}}$.

Consider the following functional SPDE on \mathbb{H} :

$$\boxed{1.1} \quad (1.1) \quad dX(t) = AX(t)dt + b(t, X(t))dt + B(t, X_t)dt + Q(t, X(t))dW(t), \quad X_0 = \xi \in \mathcal{C},$$

where $(A, \mathcal{D}(A))$ is a negative definite self-adjoint operator on \mathbb{H} , $B : [0, \infty) \times \mathcal{C} \rightarrow \mathbb{H}$, $b : [0, \infty) \times \mathbb{H} \rightarrow \mathbb{H}$ are measurable and locally bounded (i.e. bounded on bounded sets), and $Q : [0, \infty) \times \mathbb{H} \rightarrow \mathcal{L}(\bar{\mathbb{H}}; \mathbb{H})$ is measurable, where $\mathcal{L}(\bar{\mathbb{H}}; \mathbb{H})$ is the space of bounded linear operators from $\bar{\mathbb{H}}$ to \mathbb{H} .

Let $\|\cdot\|$ and $\|\cdot\|_{\text{HS}}$ denote the operator norm and the Hilbert-Schmidt norm respectively, and let $\mathcal{L}_{\text{HS}}(\bar{\mathbb{H}}; \mathbb{H})$ be the space of all Hilbert-Schmidt operators from $\bar{\mathbb{H}}$ to \mathbb{H} . Let A, B and Q satisfy the following two assumptions:

- (a1)** $(-A)^{\varepsilon-1}$ is of trace class for some $\varepsilon \in (0, 1)$; i.e. $\sum_{n=1}^{\infty} \lambda_n^{\varepsilon-1} < \infty$ for $0 < \lambda_1 \leq \lambda_2 \leq \dots$ being all eigenvalues of $-A$ counting multiplicities.
- (a2)** $B \in C([0, \infty) \times \mathcal{C}; \mathbb{H})$, $Q \in C([0, \infty) \times \mathbb{H}; \mathcal{L}(\bar{\mathbb{H}}; \mathbb{H}))$ such that for every $t \geq 0$, $Q(t, \cdot) \in C^2(\mathbb{H}; \mathcal{L}(\bar{\mathbb{H}}; \mathbb{H}))$, and $(QQ^*)(t, x)$ is invertible for all $(t, x) \in [0, \infty) \times \mathbb{H}$. Moreover,

$$\|[\nabla B(t, \cdot)](\xi)\| + \sum_{j=0}^2 \|[\nabla^j Q(t, \cdot)](\xi(0))\| + \|(QQ^*)^{-1}(t, \xi(0))\|$$

is locally bounded in $(t, \xi) \in [0, \infty) \times \mathcal{C}$, where $\|[\nabla B(t, \cdot)](\xi)\|$ stands for the local Lipschitz constant of $B(t, \cdot)$ at ξ .

Next, to describe the singularity of b , we introduce

$$\mathcal{D} = \left\{ \phi : [0, \infty) \rightarrow [0, \infty) \text{ is increasing, } \phi^2 \text{ is concave, } \int_0^1 \frac{\phi(s)}{s} ds < \infty \right\}$$

and

$$\mathcal{A} = \left\{ a \in \mathcal{B}((0, \infty); (0, \infty)), \int_0^1 \sup_{i \geq 1} \frac{\lambda_i e^{-\lambda_i s}}{a(\lambda_i)} ds < \infty \right\}.$$

Remark 1.1 The condition $\int_0^1 \frac{\phi(s)}{s} ds < \infty$ is well known as Dini condition, due to the notion of Dini continuity. Obviously the class \mathcal{D} contains $\phi(s) := \frac{K}{\log^{1+\delta}(c+s^{-1})}$ for constants $K, \delta > 0$ and large enough $c \geq e$ such that ϕ^2 is concave.

Remark 1.2 For any $a \in \mathcal{A}$, we have $\lim_{i \rightarrow \infty} a(\lambda_i) = \infty$. The class \mathcal{A} contains a lot of functions. Next, we give a class \mathcal{A}' also containing many functions but the condition in it is more easily to check than the one in \mathcal{A} . Letting

$$\mathcal{A}' = \left\{ a \in \mathcal{B}((0, \infty); (0, \infty)), a, \frac{x}{a(x)} \text{ are non-decreasing, } \int_1^\infty \frac{1}{sa(s)} ds < \infty \right\},$$

we claim $\mathcal{A}' \subset \mathcal{A}$.

Proof. For any $a \in \mathcal{A}'$, $s \in (0, 1)$, we have

$$\sup_{x \geq \frac{1}{s}} \frac{x}{a(x)} e^{-xs} \leq \sup_{x \geq \frac{1}{s}} \frac{x}{a(\frac{1}{s})} e^{-xs} \leq \frac{\frac{1}{s}}{a(\frac{1}{s})} e^{-1} \leq \frac{\frac{1}{s}}{a(\frac{1}{s})}.$$

On the other hand,

$$\sup_{1 \wedge \lambda_1 \leq x < \frac{1}{s}} \frac{x}{a(x)} e^{-xs} \leq \sup_{1 \wedge \lambda_1 \geq x < \frac{1}{s}} \frac{x}{a(x)} \leq \frac{\frac{1}{s}}{a(\frac{1}{s})}.$$

So

$$\int_0^1 \sup_{i \geq 1} \frac{\lambda_i e^{-\lambda_i s}}{a(\lambda_i)} ds \leq \int_0^1 \sup_{x \in [1 \wedge \lambda_1, \infty)} \frac{x}{a(x)} e^{-xs} ds \leq \int_0^1 \frac{\frac{1}{s}}{a(\frac{1}{s})} ds = \int_1^\infty \frac{1}{sa(s)} ds < \infty.$$

This means $a \in \mathcal{A}$, i.e. $\mathcal{A}' \subset \mathcal{A}$. □

Finally, we give some functions which belong to \mathcal{A} .

- (i) $a(x) := x^\delta$ for any $\delta \in (0, 1]$;
- (ii) $a(x) := \log^{1+\delta}(c+x)$ for $\delta > 0$ and $c \geq e^{1+\delta}$;
- (iii) $a(x) \geq a_1(x)$, $a_1 \in \mathcal{A}'$, for instance, $a(x) = x^\delta(\sin x + 2)$, $\delta \in (0, 1]$.

(i) and (ii) are in \mathcal{A}' . As to (iii), note the fact that if $a \in \mathcal{A}$, then $\tilde{a} \in \mathcal{B}((0, \infty); (0, \infty))$ satisfying $\tilde{a}(x) \geq a(x)$, $x \geq R_0$ for some constant R_0 is also in \mathcal{A} .

Next, for any $a \in \mathcal{B}((0, \infty); (0, \infty))$, let $\mathbb{H}_a = \{x \in \mathbb{H}, |a(-A)x| < \infty\}$ equipped the norm $\|x\|_a := |a(-A)x|$, $x \in \mathbb{H}_a$. Then $(\mathbb{H}_a, \|\cdot\|_a)$ is a Banach space. Note that $\mathbb{H}_1 = \mathbb{H}$. To obtain the pathwise uniqueness of (1.1), we shall need the following condition.

(a3) For any $(t, x) \in [0, \infty) \times \mathbb{H}$,

$$\boxed{1.2} \quad (1.2) \quad \lim_{n \rightarrow \infty} \|Q(t, x) - Q(t, \pi_n x)\|_{\text{HS}}^2 := \lim_{n \rightarrow \infty} \sum_{k \geq 1} |[Q(t, x) - Q(t, \pi_n x)] \bar{e}_k|^2 = 0.$$

Moreover, there exists a function $a \in \mathcal{A}$ such that $b : [0, \infty) \times \mathbb{H} \rightarrow \mathbb{H}_a$ is measurable and locally bounded, and for any $n \geq 1$, there exists $\phi_n \in \mathcal{D}$ such that

$$\boxed{1.3} \quad (1.3) \quad |b(t, x) - b(t, y)| \leq \phi_n(|x - y|), \quad t \in [0, n], x, y \in \mathbb{H}, |x| \vee |y| \leq n.$$

Remark 1.3 See Remark 3.1 for the reason why we replace [1, (a3)] with the present (a3).

In general, the mild solution (if exists) to (1.1) can be explosive, so we consider mild solutions with life time.

Definition 1.1. A continuous adapted \mathcal{C} -valued process $(X_t)_{t \in [0, \zeta]}$ is called a mild solution to (1.1) with life time ζ , if $\zeta > 0$ is a stopping time such that \mathbb{P} -a.s $\limsup_{t \uparrow \zeta} |X(t)| = \infty$ holds on $\{\zeta < \infty\}$, and \mathbb{P} -a.s

$$\begin{aligned} X(t) &= e^{A(t \vee 0)} X(t \wedge 0) + \int_0^{t \vee 0} e^{A(t-s)} (b(s, X(s)) + B(s, X_s)) ds \\ &\quad + \int_0^{t \vee 0} e^{A(t-s)} Q(s, X(s)) dW(s), \quad t \in [-r, \zeta) \cap (-\infty, \zeta). \end{aligned}$$

The following lemma is a crucial tool in the proof of our results, see [3, Proposition 7.9].

L1.1 **Lemma 1.1.** Let $\{S(t)\}_{t \geq 0}$ be a C_0 -contractive semigroup on \mathbb{H} . Assume there exists $\alpha \in (0, \frac{1}{2})$ and $s > 0$ such that

$$\text{1.4} \quad \int_0^s t^{-2\alpha} \|S(t)\|_{\text{HS}}^2 dt < \infty.$$

Then for every $q \in (1, \frac{1}{2\alpha})$, $T > 0$, there exists $c_q > 0$ such that for any $\mathcal{L}(\bar{\mathbb{H}}; \mathbb{H})$ -valued predictable process Φ , there exists a continuous version of $\int_0^\cdot S(\cdot - s) \Phi(s) dW(s)$ such that

$$\begin{aligned} \text{1.5} \quad \mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t S(t-s) \Phi(s) dW(s) \right|^{2q} \right] &\leq c_q \left[\int_0^T t^{-2\alpha} \|S(t)\|_{\text{HS}}^2 dt \right]^q \\ &\quad \times \mathbb{E} \left[\int_0^T \|\Phi(t)\|^{2q} dt \right]. \end{aligned}$$

Remark 1.4 Note that (a1) implies (1.4) for $\alpha = \frac{\varepsilon}{2}$ by a simple calculus:

$$\begin{aligned} \int_0^s t^{-2\alpha} \|S(t)\|_{\text{HS}}^2 dt &= \sum_{i=1}^{\infty} \int_0^s t^{-2\alpha} e^{-2\lambda_i t} dt \\ &\leq \sum_{i=1}^{\infty} \lambda_i^{2\alpha-1} \int_0^{\infty} u^{-2\alpha} e^{-2u} du \leq C \sum_{i=1}^{\infty} \lambda_i^{2\alpha-1} < \infty. \end{aligned}$$

2 Main results

T2.1 **Theorem 2.1.** Assume (a1), (a2) and (a3).

(1) The equation (1.1) has a unique mild solution $(X_t)_{t \in [0, \zeta]}$ with life time ζ .

- (2) Let $\|Q(t)\|_\infty := \sup_{x \in \mathbb{H}} \|Q(t, x)\|$ be locally bounded in $t \geq 0$. If there exist two positive increasing functions $\Phi, h : [0, \infty) \times [0, \infty) \rightarrow (0, \infty)$ such that $\int_1^\infty \frac{ds}{\Phi_t(s)} = \infty$ for any $t > 0$ and

2.1 (2.1) $\langle B(t, \xi + \eta) + b(t, (\xi + \eta)(0)), \xi(0) \rangle \leq \Phi_t(\|\xi\|_\infty^2) + h_t(\|\eta\|_\infty), \quad \xi, \eta \in \mathcal{C}, t \geq 0,$
then the mild solution is non-explosive.

For simplicity, we introduced some notations. For any $a \in \mathcal{B}([0, \infty), (0, \infty))$ and \mathbb{H}_a -valued function f on $[0, T] \times \mathbb{H}$, let

$$\|f\|_{T, \infty, a} = \sup_{t \in [0, T], x \in \mathbb{H}} |a(-A)f(t, x)|$$

Similarly if f is a operator-valued (for example $\mathcal{L}(\mathbb{H}, \mathbb{H}_a)$) map defined on $[0, T] \times \mathbb{H}$, let

$$\|f\|_{T, \infty, a} = \sup_{t \in [0, T], x \in \mathbb{H}} \|a(-A)f(t, x)\|.$$

If $a = 1$, we omit it.

In order to apply Zvonkin type transformation, we need the following global conditions:

- (a2') B, Q satisfy (a2), and there exists a positive increasing function $C_{B, Q} : [0, \infty) \rightarrow (0, \infty)$ such that

$$\|\nabla B\|_{T, \infty} + \sum_{j=0}^2 \|\nabla^j Q\|_{T, \infty} + \|(QQ^*)^{-1}\|_{T, \infty} < C_{B, Q}(T), \quad T \geq 0.$$

- (a3') Q satisfies (1.2). In addition, for any $T > 0$, there exist $a \in \mathcal{A}$ and $\phi \in \mathcal{D}$ such that

2.2 (2.2) $\|b\|_{T, \infty, a} < \infty,$

and

2.3 (2.3) $|b(t, x) - b(t, y)| \leq \phi(|x - y|), \quad t \in [0, T], x, y \in \mathbb{H}.$

According to Theorem 2.1, under (a1), (a2') and (a3'), the unique mild solution X_t^ξ of (1.1) is non-explosive. Fixing $r < \infty$, the associated Markov semigroup P_t of X_t^ξ is defined as

$$P_t f(\xi) = \mathbb{E} f(X_t^\xi), \quad f \in \mathcal{B}_b(\mathcal{C}), t \geq 0, \xi \in \mathcal{C}.$$

To derive the Harnack inequalities for P_t with $t > r$, we need a stronger condition (a3'') in stead of (a3') as follows:

- (a3'') Q satisfies (1.2). In addition, for any $T > 0$,

2.4 (2.4) $\left\| (-A)^{\frac{1}{2}} b \right\|_{T, \infty} < \infty,$

and there exists $\phi \in \mathcal{D}$ such that

2.5 (2.5) $\left| (-A)^{\frac{1-\varepsilon}{2}} [b(t, x) - b(t, y)] \right| \leq \phi(|x - y|), \quad t \in [0, T], x, y \in \mathbb{H},$

where ε is in (a1).

Then we have

T2.2 **Theorem 2.2.** *Assume $(\mathbf{a1})$, $(\mathbf{a2}')$, $(\mathbf{a3}'')$. In addition, if $\|B(t)\|_\infty := \sup_{\xi \in \mathcal{C}} |B(t, \xi)|$ is locally bounded in $t \geq 0$, and for any $T > 0$, there exists a constant $C(T) > 0$ such that*

2.6 (2.6) $\|Q(t, x) - Q(t, y)\|_{\text{HS}}^2 \leq C(T)|x - y|^2, \quad t \in [0, T], x, y \in \mathbb{H}.$

Then for every $T > r$ and positive function $f \in \mathcal{B}_b(\mathcal{C})$,

(1) the log-Harnack inequality holds, i.e.

2.7 (2.7) $P_T \log f(\eta) \leq \log P_T f(\xi) + H(T, \xi, \eta), \quad \xi, \eta \in \mathcal{C}$

with

$$H(T, \xi, \eta) = C \left(\frac{|\xi(0) - \eta(0)|^2}{T - r} + \|\xi - \eta\|_\infty^2 \right)$$

for some constant $C > 0$.

(2) There exists $K > 0$ such that for any $p > (1 + K)^2$, the Harnack inequality with power

2.8 (2.8) $P_T f(\eta) \leq (P_T f^p(\xi))^{\frac{1}{p}} \exp \Psi_p(T; \xi, \eta), \quad \xi, \eta \in \mathcal{C}$

holds, where

$$\Psi_p(T; \xi, \eta) = C(p) \left\{ 1 + \frac{|\xi(0) - \eta(0)|^2}{T - r} + \|\xi - \eta\|_\infty^2 \right\}$$

for a decreasing function $C : ((1 + K)^2, \infty) \rightarrow (0, \infty)$.

The reminder of the paper is organized as follows: In Section 3, we prove the pathwise uniqueness, in Section 4, combining Section 3 with a truncating argument, we prove Theorem 2.1, in Section 5, we investigate the Harnack inequalities for the semigroup by finite-dimensional approximations. Results in Sections 3 and 5 are derived under some global conditions, and assertions in Section 4 are derived under some local conditions.

3 Pathwise uniqueness

In this section, we transform (1.1) to a regular equation and investigate the pathwise uniqueness of it, which is equivalent to that of (1.1). To this end, firstly, we consider the gradient estimate for the following SPDE (3.1), which is crucial in the proof of the regularity of the solution to the equation (3.3), see [1] for details. Differently, we need a modified gradient estimate in the present case, and we will give a proof in detail.

3.1 (3.1) $dZ_{s,t}^x = AZ_{s,t}^x dt + Q(t, Z_{s,t}^x) dW(t), \quad Z_{s,s}^x = x, t \geq s \geq 0.$

Then under $(\mathbf{a1})$, $(\mathbf{a2}')$ with $B = 0$, (3.1) has a unique mild solution $\{Z_{s,t}^x\}_{t \geq s}$. Let $P_{s,t}^0$ be the associated Markov semigroup.

Firstly, for any $f \in \mathcal{B}_b(\mathbb{H}, \mathbb{H})$, $\eta, x \in \mathbb{H}$, $0 \leq s < t \leq T$, by [1, (2.12)], we have

$$\sum_{i=1}^{\infty} |\nabla_{\eta} P_{s,t}^0 \langle f, e_i \rangle(x)|^2 \leq \sum_{i=1}^{\infty} \frac{c}{t-s} P_{s,t}^0 |\langle f, e_i \rangle|^2(x) |\eta|^2 = \frac{c}{t-s} P_{s,t}^0 |f|^2(x) |\eta|^2.$$

Then $\mathbb{H} \ni \nabla_{\eta} P_{s,t}^0 f(x) \left(:= \sum_{i=1}^{\infty} (\nabla_{\eta} P_{s,t}^0 \langle f, e_i \rangle(x)) e_i \right)$ and

$$\boxed{3.2} \quad (3.2) \quad |\nabla P_{s,t}^0 f(x)|^2 \leq \frac{c}{t-s} P_{s,t}^0 |f|^2(x)$$

for a constant $c > 0$.

Next, by (3.2), for any $a \in \mathcal{B}((0, \infty); (0, \infty))$, $f \in \mathcal{B}_b(\mathbb{H}, \mathbb{H}_a)$ and $x, \eta, \eta' \in \mathbb{H}$, $0 \leq s < t \leq T$, it holds that

$$\begin{aligned} \sum_{i=1}^{\infty} a(\lambda_i)^2 (\nabla_{\eta} P_{s,t}^0 \langle f, e_i \rangle(x))^2 &= \sum_{i=1}^{\infty} (\nabla_{\eta} P_{s,t}^0 \langle a(-A)f, e_i \rangle(x))^2 \\ &\leq \frac{c}{t-s} P_{s,t}^0 |a(-A)f|^2(x) |\eta|^2 < \infty. \end{aligned}$$

Then $\nabla_{\eta} P_{s,t}^0 f(x)$ belong to the domain of $a(-A)$ and

$$\begin{aligned} a(-A) \nabla_{\eta} P_{s,t}^0 f(x) &= \sum_{i=1}^{\infty} a(\lambda_i) \nabla_{\eta} P_{s,t}^0 \langle f, e_i \rangle(x) e_i \\ &= \sum_{i=1}^{\infty} \nabla_{\eta} P_{s,t}^0 \langle a(-A)f, e_i \rangle(x) e_i \\ &= \nabla_{\eta} P_{s,t}^0 (a(-A)f)(x). \end{aligned}$$

Similarly, according to [1, (2.16)], it is easy to see that

$$\boxed{3.3} \quad (3.3) \quad a(-A) \nabla_{\eta'} \nabla_{\eta} P_{s,t}^0 f(x) = \nabla_{\eta'} \nabla_{\eta} P_{s,t}^0 (a(-A)f)(x).$$

In a word,

$$\boxed{3.4} \quad (3.4) \quad a(-A) \nabla^k P_{s,t}^0 f = \nabla^k P_{s,t}^0 (a(-A)f), \quad f \in \mathcal{B}_b(\mathbb{H}, \mathbb{H}_a), \quad 0 \leq s < t \leq T, \quad k = 0, 1, 2.$$

By [1], to obtain the pathwise uniqueness of (1.1), we need to study the following equation:

$$\boxed{3.5} \quad (3.5) \quad u(s, x) = \int_s^T e^{-\lambda(t-s)} P_{s,t}^0 (\nabla_{b(t, \cdot)} u(t, \cdot) + b(t, \cdot))(x) dt, \quad s \in [0, T].$$

The following Lemma is a modified result of [1, Lemma 2.3].

L3.1 **Lemma 3.1.** *Assume **(a1)**, **(a2')** with $B = 0$, (2.2). Let $T > 0$ be fixed. Then there exists a constant $\lambda(T) > 0$ such that the following assertions hold.*

(1) *For any $\lambda \geq \lambda(T)$, the equation (3.5) has a unique solution $u \in C([0, T]; C_b^1(\mathbb{H}; \mathbb{H}_a))$.*

(2) If moreover (2.3) holds, then we have

$$\boxed{3.6} \quad (3.6) \quad \lim_{\lambda \rightarrow \infty} \|u\|_{T,\infty,a} + \|\nabla u\|_{T,\infty,a} + \|\nabla^2 u\|_{T,\infty} = 0.$$

Proof. (1) Let $\mathcal{H} = C([0, T]; C_b^1(\mathbb{H}; \mathbb{H}_a))$, which is a Banach space under the norm

$$\begin{aligned} \|u\|_{\mathcal{H}} &:= \|u\|_{T,\infty,a} + \|\nabla u\|_{T,\infty,a} \\ &= \sup_{t \in [0, T], x \in \mathbb{H}} |a(-A)u(t, x)| + \sup_{t \in [0, T], x \in \mathbb{H}} \|a(-A)\nabla u(t, x)\|, \quad u \in \mathcal{H}. \end{aligned}$$

For any $u \in \mathcal{H}$, define

$$(\Gamma u)(s, x) = \int_s^T e^{-\lambda(t-s)} P_{s,t}^0 (\nabla_{b(t,\cdot)} u(t, \cdot) + b(t, \cdot))(x) dt, \quad s \in [0, T].$$

Then we have $\Gamma \mathcal{H} \subset \mathcal{H}$. In fact, for any $u \in \mathcal{H}$, by **(a2')**, (2.2), (3.4) and dominated convergence theorem, it holds that

$$\begin{aligned} \|\Gamma u\|_{T,\infty,a} &= \sup_{s \in [0, T], x \in \mathbb{H}} \left| \int_s^T e^{-\lambda(t-s)} P_{s,t}^0 (a(-A)\nabla_{b(t,\cdot)} u(t, \cdot) + a(-A)b(t, \cdot))(x) dt \right| \\ &\leq \sup_{s \in [0, T]} \int_s^T e^{-\lambda(t-s)} (\|b\|_{T,\infty} \|\nabla u\|_{T,\infty,a} + \|b\|_{T,\infty,a}) dt \\ &\leq (\|b\|_{T,\infty} \|\nabla u\|_{T,\infty,a} + \|b\|_{T,\infty,a}) \int_0^T e^{-\lambda t} dt \\ &\leq \frac{\|b\|_{T,\infty} \|\nabla u\|_{T,\infty,a} + \|b\|_{T,\infty,a}}{\lambda} < \infty. \end{aligned}$$

Again by **(a2')**, (2.2), (3.4) and dominated convergence theorem, we have

$$\begin{aligned} \|\nabla \Gamma u\|_{T,\infty,a} &= \sup_{s \in [0, T], x \in \mathbb{H}, |\eta| \leq 1} \left\| \int_s^T e^{-\lambda(t-s)} \nabla_{\eta} P_{s,t}^0 (a(-A)\nabla_{b(t,\cdot)} u(t, \cdot) + a(-A)b(t, \cdot))(x) dt \right\| \\ &\leq C \sup_{s \in [0, T]} \int_s^T \frac{e^{-\lambda(t-s)}}{\sqrt{t-s}} (\|b\|_{T,\infty} \|\nabla u\|_{T,\infty,a} + \|b\|_{T,\infty,a}) dt \\ &\leq C (\|b\|_{T,\infty} \|\nabla u\|_{T,\infty,a} + \|b\|_{T,\infty,a}) \int_0^T \frac{e^{-\lambda t}}{\sqrt{t}} dt \\ &\leq C \frac{\|b\|_{T,\infty} \|\nabla u\|_{T,\infty,a} + \|b\|_{T,\infty,a}}{\sqrt{\lambda}} < \infty. \end{aligned}$$

So, $\Gamma \mathcal{H} \subset \mathcal{H}$. Next, by the fixed-point theorem, it suffices to show that for large enough $\lambda > 0$, Γ is contractive on \mathcal{H} . To do this, for any $u, \tilde{u} \in \mathcal{H}$, it is easy to see that

$$\begin{aligned} \|\Gamma u - \Gamma \tilde{u}\|_{T,\infty,a} &\leq \frac{\|b\|_{T,\infty}}{\lambda} \|\nabla u - \nabla \tilde{u}\|_{T,\infty,a}, \\ \|\nabla(\Gamma u - \Gamma \tilde{u})\|_{T,\infty,a} &\leq C \frac{\|b\|_{T,\infty}}{\sqrt{\lambda}} \|\nabla u - \nabla \tilde{u}\|_{T,\infty,a}. \end{aligned}$$

So we can find $\lambda(T) > 0$ such that Γ is contractive on \mathcal{H} with $\lambda > \lambda(T)$, thus we prove (1).

(2) Combining the proof of (1) and the proof of [1, Lemma 2.3 (2)], it is easy to obtain (2). Here to save space, we do not repeat the process. \square

The next Lemma gives a regular representation of (1.1). See the proof of [1, Proposition 2.5] for details.

L3.2 **Lemma 3.2.** *Assume $(\mathbf{a1})$, $(\mathbf{a2}')$ and $(\mathbf{a3}')$. Then for any $T > 0$, there exists a constant $\lambda(T) > 0$ such that for any stopping time τ , any adapted continuous \mathcal{C} -valued process $(X_t)_{t \in [0, T \wedge \tau]}$ with \mathbb{P} -a.s.*

$$\begin{aligned} X(t) &= e^{At}X(0) + \int_0^t e^{A(t-s)}(b(s, X(s)) + B(s, X_s))ds \\ &\quad + \int_0^t e^{A(t-s)}Q(s, X(s))dW(s), \quad t \in [0, \tau \wedge T], \end{aligned}$$

and any $\lambda \geq \lambda(T)$, there holds

$$\begin{aligned} X(t) &= e^{At}[X(0) + u(0, X(0))] - u(t, X(t)) \\ &\quad + \int_0^t (\lambda - A)e^{A(t-s)}u(s, X(s))ds \\ (3.7) \quad &\quad + \int_0^t e^{A(t-s)}[I + \nabla u(s, X(s))]B(s, X_s)ds \\ &\quad + \int_0^t e^{A(t-s)}[I + \nabla u(s, X(s))]Q(s, X(s))dW(s), \quad t \in [0, \tau \wedge T], \end{aligned}$$

where u solves (3.5), and $\nabla u(s, z)v := [\nabla_v u(s, \cdot)](z)$ for $v, z \in \mathbb{H}$.

Remark 3.1 The second term on the right side of (3.7) has the same form with the neutral functional SPDE, see [4]. In the case without delay, this can be dealt with by Fubini Theorem, see [1] for details. However, due to the delay, Fubini Theorem is unavailable in the present case. Instead, to prove the pathwise uniqueness, we need a condition like [4, (H3)], which can be ensured by Lemma 3.1 and (2.2), see the proof of the following Proposition 3.3.

Now, we present a complete proof of the pathwise uniqueness to (1.1).

P3.3 **Proposition 3.3.** *Assume $(\mathbf{a1})$, $(\mathbf{a2}')$ and $(\mathbf{a3}')$. Let $\{X_t\}_{t \geq 0}, \{Y_t\}_{t \geq 0}$ be two adapted continuous \mathcal{C} -valued processes with $X_0 = Y_0 = \xi \in \mathcal{C}$. For any $n \geq 1$, let*

$$\tau_n^X = n \wedge \inf\{t \geq 0 : |X(t)| \geq n\}, \quad \tau_n^Y = n \wedge \inf\{t \geq 0 : |Y(t)| \geq n\}.$$

If \mathbb{P} -a.s. for all $t \in [0, \tau_n^X \wedge \tau_n^Y]$, there holds :

$$\begin{aligned} X(t) &= e^{At}\xi(0) + \int_0^t e^{A(t-s)}(b(s, X(s)) + B(s, X_s))ds + \int_0^t e^{A(t-s)}Q(s, X(s))dW(s), \\ Y(t) &= e^{At}\xi(0) + \int_0^t e^{A(t-s)}(b(s, Y(s)) + B(s, Y_s))ds + \int_0^t e^{A(t-s)}Q(s, Y(s))dW(s), \end{aligned}$$

then \mathbb{P} -a.s. $X(t) = Y(t)$, for all $t \in [0, \tau_n^X \wedge \tau_n^Y]$. In particular, \mathbb{P} -a.s. $\tau_n^X = \tau_n^Y$.

Proof. For any $n \geq 1$, let $\tau_n := \tau_n^X \wedge \tau_n^Y$. It suffices to prove that for any $T > 0$,

$$\boxed{3.8} \quad (3.8) \quad \mathbb{E} \sup_{s \in [0, T]} |X(s \wedge \tau_n) - Y(s \wedge \tau_n)|^{2p} = 0.$$

holds for some $p > 1$. In the following, we fix $T > 0$ and $p > 1$. Taking λ large enough such that assertions in Lemma 3.1, Lemma 3.2 hold and

$$\boxed{3.9} \quad (3.9) \quad \frac{5^{4p-1}}{2^{2p+1}} \left(\|a(-A)\nabla u(t, \cdot)\|_\infty \int_0^T \|(-A)[a(-A)]^{-1}e^{As}\| ds \right)^{2p} + \|\nabla u(t, \cdot)\|_\infty \leq \frac{1}{5}$$

for any $t \in [0, T]$. By (3.7) for $\tau = \tau_n$, we have \mathbb{P} -a.s. for any $t \in [0, \tau_n \wedge T]$,

$$\begin{aligned} & [X(t) + u(t, X(t))] - [Y(t) + u(t, Y(t))] \\ &= \int_0^t (\lambda - A)e^{A(t-s)}[u(s, X(s)) - u(s, Y(s))]ds \\ &+ \int_0^t e^{A(t-s)}\{[I + \nabla u(s, X(s))]B(s, X_s) - [I + \nabla u(s, Y(s))]B(s, Y_s)\}ds \\ &+ \int_0^t e^{A(t-s)}\{[I + \nabla u(s, X(s))]Q(s, X(s)) - [I + \nabla u(s, Y(s))]Q(s, Y(s))\}dW(s). \end{aligned}$$

Then (3.9) yields that

$$\begin{aligned} & \mathbb{E} \sup_{t \in [0, q]} |X(t \wedge \tau_n) - Y(t \wedge \tau_n)|^{2p} \\ & \leq \frac{5^{4p-1}}{4^{2p}} \mathbb{E} \sup_{t \in [0, q]} \left| \int_0^{t \wedge \tau_n} (\lambda - A)e^{A(t-s)}[u(s, X(s)) - u(s, Y(s))]ds \right|^{2p} \\ & + \frac{5^{4p-1}}{4^{2p}} \mathbb{E} \sup_{t \in [0, q]} \left| \int_0^{t \wedge \tau_n} e^{A(t-s)}[I + \nabla u(s, X(s))][B(s, X_s) - B(s, Y_s)]ds \right|^{2p} \\ \boxed{3.10} \quad (3.10) \quad & + \frac{5^{4p-1}}{4^{2p}} \mathbb{E} \sup_{t \in [0, q]} \left| \int_0^{t \wedge \tau_n} e^{A(t-s)}[\nabla u(s, X(s)) - \nabla u(s, Y(s))]B(s, Y_s)ds \right|^{2p} \\ & + \frac{5^{4p-1}}{4^{2p}} \mathbb{E} \sup_{t \in [0, q]} \left| \int_0^{t \wedge \tau_n} e^{A(t-s)}[\nabla u(s, X(s)) - \nabla u(s, Y(s))]Q(s, X(s))dW(s) \right|^{2p} \\ & + \frac{5^{4p-1}}{4^{2p}} \mathbb{E} \sup_{t \in [0, q]} \left| \int_0^{t \wedge \tau_n} e^{A(t-s)}(I + \nabla u(s, Y(s)))[Q(s, X(s)) - Q(s, Y(s))]dW(s) \right|^{2p} \\ & =: I_1 + I_2 + I_3 + I_4 + I_5, \quad q \in [0, T]. \end{aligned}$$

Firstly, let

$$\eta_q = \mathbb{E} \sup_{t \in [0, q]} |X(t \wedge \tau_n) - Y(t \wedge \tau_n)|^{2p}.$$

Moreover, by (3.9), there exists a constant $C(p, \lambda, T) > 0$ such that

$$\begin{aligned}
I_1 &\leq C(p, \lambda, T) \int_0^q \eta_s ds \\
&+ \left[\frac{5^{4p-1}}{2^{2p+1}} \left(\|a(-A) \nabla u(t, \cdot)\|_\infty \int_0^T \|(-A)[a(-A)]^{-1} e^{As}\| ds \right)^{2p} \right] \eta_q \\
&\leq C(p, \lambda, T) \int_0^q \eta_s ds + \frac{1}{5} \eta_q.
\end{aligned}
\tag{3.11}$$

Since A is negative definite, by (3.6), $(\mathbf{a2}')$, the same initial value of X and Y , and Hölder inequality, it holds that

$$\begin{aligned}
I_2 &\leq C \mathbb{E} \int_0^{q \wedge \tau_n} |B(s, X_s) - B(s, Y_s)|^{2p} ds \\
&\leq C_1 \mathbb{E} \int_0^q \sup_{t \in [0, s]} |X(t \wedge \tau_n) - Y(t \wedge \tau_n)|^{2p} ds \\
&\leq C_1 \int_0^q \eta_s ds
\end{aligned}
\tag{3.12}$$

for a constant $C_1 > 0$. Similarly, Combining (3.6) and the local boundedness of B , we obtain

$$I_3 \leq C_2 \int_0^q \eta_s ds \tag{3.13}$$

for a constant $C_2 > 0$.

Next, in view of $(\mathbf{a2}')$, Lemma 1.1 and Remark 1.4, we have

$$I_4 + I_5 \leq C_3 \mathbb{E} \int_0^{q \wedge \tau_n} |X(s) - Y(s)|^{2p} ds = C_3 \int_0^q \eta_s ds, \tag{3.14}$$

for a constant $C_3 > 0$. Combining (3.10), (3.11), (3.12), (3.13) and (3.14), there exists a constant C_0 such that

$$\eta_l \leq \frac{1}{5} \eta_l + C_0 \int_0^l \eta_q dq, \quad l \in [0, T].$$

By Gronwall's inequality, we obtain $\eta_T = 0$, i.e. (3.8) holds. \square

4 Proof of Theorem 2.1

Proof of Theorem 2.1. (a) We first assume that $(\mathbf{a1})$, $(\mathbf{a2}')$ and $(\mathbf{a3}')$ hold. Consider the following SPDE on \mathbb{H} :

$$dZ^\xi(t) = AZ^\xi(t)dt + Q(t, Z^\xi(t))dW(t), \quad Z^\xi(0) = \xi(0).$$

It is easy to see that the above equation has a uniqueness non-explosive mild solution:

$$Z^\xi(t) = e^{At}\xi(0) + \int_0^t e^{A(t-s)}Q(s, Z^\xi(s))dW(s), \quad t \geq 0.$$

Letting $Z_0^\xi = \xi$ (i.e. $Z^\xi(\theta) = \xi(\theta)$ for $\theta \in [-r, 0]$), and taking

$$\begin{aligned} W^\xi(t) &= W(t) - \int_0^t \psi(s)ds, \\ \psi(s) &= \{Q^*(QQ^*)^{-1}\}(s, Z^\xi(s))\{b(s, Z^\xi(s)) + B(s, Z_s^\xi)\}, \quad s, t \in [0, T], \end{aligned}$$

we have

$$\begin{aligned} Z^\xi(t) &= e^{At}\xi(0) + \int_0^t e^{A(t-s)}B(s, Z_s^\xi)ds \\ &\quad + \int_0^t e^{A(t-s)}b(s, Z^\xi(s))ds + \int_0^t e^{A(t-s)}Q(s, Z^\xi(s))dW^\xi(s), \quad t \in [0, T]. \end{aligned}$$

By the local boundedness of B , Girsanov theorem implies $\{W^\xi(t)\}_{t \in [0, T]}$ is a cylindrical Brownian motion on $\bar{\mathbb{H}}$ under probability $d\mathbb{Q}^\xi = R^\xi d\mathbb{P}$, where

$$R^\xi := \exp \left[\int_0^T \langle \psi(s), dW(s) \rangle_{\bar{\mathbb{H}}} - \frac{1}{2} \int_0^T |\psi(s)|_{\bar{\mathbb{H}}}^2 ds \right].$$

Then, under the probability \mathbb{Q}^ξ , $(Z^\xi(t), W^\xi(t))_{t \in [0, T]}$ is a weak mild solution to (1.1). On the other hand, by Proposition 3.3, the pathwise uniqueness holds for the mild solution to (1.1). So, by the Yamada-Watanabe principle, the equation (1.1) has a unique mild solution. Moreover, in this case the solution is non-explosive.

(b) In general, take $\psi \in C_b^\infty([0, \infty))$ such that $0 \leq \psi \leq 1$, $\psi(u) = 1$ for $u \in [0, 1]$ and $\psi(u) = 0$ for $u \in [2, \infty]$. For any $m \geq 1$, let

$$\begin{aligned} b^{[m]}(t, z) &= b(t \wedge m, z)\psi(|z|/m), \quad (t, z) \in [0, \infty) \times \mathbb{H}, \\ B^{[m]}(t, \xi) &= B(t \wedge m, \xi)\psi(\|\xi\|_\infty/m), \quad (t, \xi) \in [0, \infty) \times \mathcal{C}, \\ Q^{[m]}(t, z) &= Q(t \wedge m, z)\psi(|z|/m), \quad (t, z) \in [0, \infty) \times \mathbb{H}. \end{aligned}$$

By **(a2)** and **(a3)**, we know $B^{[m]}$, $Q^{[m]}$ and $b^{[m]}$ satisfy **(a2')**, **(a3')**. Then by (a), (1.1) for $B^{[m]}$, $Q^{[m]}$ and $b^{[m]}$ in place of B , Q , b has a unique mild solution $X^{[m]}(t)$ starting at X_0 which is non-explosive. Let

$$\zeta_0 = 0, \quad \zeta_m = m \wedge \inf\{t \geq 0 : |X^{[m]}(t)| \geq m\}, \quad m \geq 1.$$

Since $B^{[m]}(s, \xi) = B(s, \xi)$, $Q^{[m]}(s, \xi(0)) = Q(s, \xi(0))$ and $b^{[m]}(s, \xi(0)) = b(s, \xi(0))$ hold for $s \leq m$, and $\|\xi\|_\infty \leq m$, by Proposition 3.3, for any $n, m \geq 1$, we have $X^{[m]}(t) = X^{[n]}(t)$ for $t \in [0, \zeta_m \wedge \zeta_n]$. In particular, ζ_m is increasing in m . Let $\zeta = \lim_{m \rightarrow \infty} \zeta_m$ and

$$X(t) = \sum_{m=1}^{\infty} 1_{[\zeta_{m-1}, \zeta_m)} X^{[m]}(t), \quad t \in [0, \zeta).$$

Then it is easy to see that $X(t)_{t \in [0, \zeta]}$ is a mild solution to (1.1) with lifetime ζ and, due to Proposition 3.3, the mild solution is unique. So we prove Theorem 2.1 (1).

(c) Next, we prove the non-explosion.

Let $\|Q\|_{T, \infty} < \infty$ for $T > 0$, and let Φ, h satisfy (2.1). Let $X(t)_{t \in [0, \zeta]}$ be the mild solution to (1.1) with lifetime ζ . Let $M(t) = \int_0^t e^{A(t-s)} Q(s, X(s)) dW(s)$, $t \in [0, \zeta]$; $M(t) = 0$, $t \in [-r, 0]$. Then M_t is an adapted continuous process on \mathbb{H} up to the lifetime ζ . It is clear that $Y(t) := X(t) - M(t)$ is the mild solution to the following equation up to ζ ,

$$dY(t) = (AY(t) + b(t, Y(t) + M(t)) + B(t, Y_t + M_t))dt, \quad Y_0 = X_0.$$

Then (2.1) implies that for any $T > 0$,

$$\begin{aligned} \boxed{4.1} \quad (4.1) \quad d|Y(t)|^2 &\leq 2\langle Y(t), b(t, Y(t) + M(t)) + B(t, Y_t + M_t) \rangle dt \\ &\leq 2(\Phi_{\zeta \wedge T}(\|Y_t\|_\infty^2) + h_{\zeta \wedge T}(\|M_t\|_\infty)) dt. \end{aligned}$$

Let

$$\boxed{4.2} \quad (4.2) \quad \Psi_T(s) = \int_1^s \frac{dr}{2\Phi_{\zeta \wedge T}(r)}, \quad \alpha_T = 2\|X_0\|_\infty^2 + 2 \int_0^{\zeta \wedge T} h_{\zeta \wedge T}(\|M_s\|_\infty) ds.$$

It follows from (4.1) that

$$\begin{aligned} \boxed{4.3} \quad (4.3) \quad \sup_{t \in [0, q]} |Y(t)|^2 &\leq |X(0)|^2 + 2 \int_0^{\zeta \wedge T} h_{\zeta \wedge T}(\|M_s\|_\infty) ds \\ &\quad + 2 \int_0^q \Phi_{\zeta \wedge T} \left(\sup_{t \in [-r, s]} |Y(t)|^2 \right) ds, \quad q \in [0, \zeta \wedge T]. \end{aligned}$$

Combining (4.2) with (4.3), we have

$$\boxed{4.4} \quad (4.4) \quad \sup_{t \in [-r, q]} |Y(t)|^2 \leq \alpha_T + 2 \int_0^q \Phi_{\zeta \wedge T} \left(\sup_{t \in [-r, s]} |Y(t)|^2 \right) ds, \quad q \in [0, \zeta \wedge T].$$

Let $Z(s) = \sup_{t \in [-r, s]} |Y(t)|^2$, $s \in [0, \zeta \wedge T]$, by Biharis's inequality, (4.4) implies

$$\boxed{4.5} \quad (4.5) \quad Z(t) \leq \Psi_T^{-1}(\Psi_T(\alpha_T) + t), \quad t \in [0, \zeta \wedge T].$$

Moreover, **(a1)**, $\|Q\|_{T, \infty} < \infty$ and Lemma 1.1 yield

$$\boxed{4.6} \quad (4.6) \quad \mathbb{E} \sup_{t \in [0, \zeta \wedge T]} |M(t)|^2 < \infty.$$

So by the definition of ζ and Y , on the set $\{\zeta < \infty\}$, we have \mathbb{P} -a.s.

$$\boxed{4.7} \quad (4.7) \quad \limsup_{t \uparrow \zeta} |Y(t)| = \limsup_{t \uparrow \zeta} |X(t)| = \infty.$$

More on the set $\zeta \leq T$, \mathbb{P} -a.s. $\alpha_T < \infty$. Combining the property of Φ and (4.7), it holds that on the set $\zeta \leq T$, \mathbb{P} -a.s.

$$\infty = \limsup_{t \uparrow \zeta} |Y(t)|^2 \leq \Psi_T^{-1}(\Psi_T(\alpha_T) + T) < \infty.$$

So for any $T > 0$, $\mathbb{P}\{\zeta \leq T\} = 0$. Note that

$$\mathbb{P}\{\zeta < \infty\} = \mathbb{P}\left(\bigcup_{m=1}^{\infty} \{\zeta \leq m\}\right) \leq \sum_{m=1}^{\infty} \mathbb{P}\{\zeta \leq m\} = 0,$$

which implies the solution of (1.1) is non-explosive. \square

5 Proof of Theorem 2.2

The idea of the proof is to transform (1.1) into an equation with regular coefficients, so that the Harnack inequalities for the new equation can be derived by coupling by change of measure and finite dimension approximation, see [10, Theorem 3.4.1, Theorem 4.3.1 and Theorem 4.3.2]. To this end, we use the regularization representation (3.7).

In this section, we fix $T > r$. Under **(a1)**, **(a2')**, **(a3')** with $a(x) = x^{\frac{1}{2}}$, by Lemma 3.1 (2) and Lemma 3.2, we take large enough $\lambda(T) > 0$ such that for any $\lambda \geq \lambda(T)$, Lemma 3.2 holds and the unique solution u to (3.5) satisfies:

$$\boxed{5.1} \quad (5.1) \quad \|\nabla^2 u\|_{T,\infty} \leq \frac{1}{8}, \quad \|(-A)^{\frac{1}{2}} \nabla u\|_{T,\infty} \leq \frac{\sqrt{\lambda_1}}{8}.$$

To treat the delay part, define $u(s, \cdot) = u(0, \cdot)$ for $s \in [-r, 0]$. Let $\theta(t, x) = x + u(t, x)$, $(t, x) \in [-r, T] \times \mathbb{H}$. By (5.1), $\{\theta(t, \cdot)\}_{t \in [-r, T]}$ is a family of diffeomorphisms on \mathbb{H} . For simplicity, we write $\theta^{-1}(t, x) = [\theta^{-1}(t, \cdot)](x)$, $(t, x) \in [-r, T] \times \mathbb{H}$. By (5.1), we have

$$\boxed{5.2} \quad (5.2) \quad \frac{7}{8} \leq \|\nabla \theta(t, x)\| \leq \frac{9}{8}, \quad \frac{8}{9} \leq \|\nabla \theta^{-1}(t, x)\| \leq \frac{8}{7}, \quad (t, x) \in [-r, T] \times \mathbb{H}.$$

where $\nabla \theta(t, x) := [\nabla \theta(t, \cdot)](x)$ and $\nabla \theta^{-1}(t, x) := [\nabla \theta^{-1}(t, \cdot)](x)$.

On the other hand, for any $t \in [0, T]$, define $\theta_t : \mathcal{C} \rightarrow \mathcal{C}$ as

$$\boxed{5.3} \quad (5.3) \quad (\theta_t(\xi))(s) = \theta(t + s, \xi(s)), \quad \xi \in \mathcal{C}, s \in [-r, 0].$$

Then $\{\theta_t\}_{t \in [0, T]}$ is a family of diffeomorphisms on \mathcal{C} . Moreover, it is easy to see that for any $t \in [0, T]$,

$$\boxed{5.4} \quad (5.4) \quad (\theta_t^{-1}(\xi))(s) = \theta^{-1}(t + s, \xi(s)), \quad \xi \in \mathcal{C}, s \in [-r, 0].$$

Furthermore, it follows from (5.3) and (5.2) that

$$\boxed{5.5} \quad (5.5) \quad \begin{aligned} \|(\nabla \theta_t)(\xi)\| &:= \limsup_{\eta \rightarrow \xi} \frac{\sup_{s \in [-r, 0]} |\theta(t + s, \eta(s)) - \theta(t + s, \xi(s))|}{\|\eta - \xi\|_{\infty}} \\ &\leq \frac{9}{8}, \quad t \in [0, T], \xi \in \mathcal{C}. \end{aligned}$$

Similarly, we have

$$\boxed{5.6} \quad (5.6) \quad \|(\nabla \theta_t^{-1})(\xi)\| \leq \frac{8}{7}, \quad t \in [0, T], \xi \in \mathcal{C}.$$

Now, letting $\{X^\xi(t)\}_{t \in [-r, T]}$ solve (1.1) with $X_0^\xi = \xi \in \mathcal{C}$, by (3.7), for any $\lambda \geq \lambda(T)$, $\{Y^\xi(t) = \theta(t, X^\xi(t))\}_{t \in [-r, T]}$ with $Y_t^\xi = \theta_t(X_t^\xi)$ satisfies

$$\begin{aligned} Y^\xi(t) &= e^{At} Y^\xi(0) + \int_0^t e^{A(t-s)} (\lambda - A) u(s, \theta^{-1}(s, Y^\xi(s))) ds \\ &+ \int_0^t e^{A(t-s)} \nabla \theta(s, \theta^{-1}(s, Y^\xi(s))) B(s, \theta_s^{-1}(Y_s^\xi)) ds \\ &+ \int_0^t e^{A(t-s)} \nabla \theta(s, \theta^{-1}(s, Y^\xi(s))) Q(s, \theta^{-1}(s, Y^\xi(s))) dW(s), \quad t \in [0, T]. \end{aligned} \quad \boxed{5.7} \quad (5.7)$$

Let

$$\boxed{5.8} \quad (5.8) \quad \bar{b}(t, x) = (\lambda - A) u(t, \theta^{-1}(t, x)), \quad t \in [0, T], x \in \mathbb{H}.$$

$$\boxed{5.9} \quad (5.9) \quad \bar{B}(t, \xi) = \nabla \theta(t, \theta^{-1}(t, \xi(0))) B(t, \theta_t^{-1}(\xi)), \quad t \in [0, T], \xi \in \mathcal{C}.$$

$$\boxed{5.10} \quad (5.10) \quad \bar{Q}(t, x) = \nabla \theta(t, \theta^{-1}(t, x)) Q(t, \theta^{-1}(t, x)), \quad t \in [0, T], x \in \mathbb{H}.$$

Then for any $\lambda \geq \lambda(T)$, $\{\bar{X}^\xi(t) := Y^{\theta_0^{-1}(\xi)}(t)\}_{t \in [-r, T]}$ is a mild solution to the equation

$$\boxed{5.11} \quad (5.11) \quad d\bar{X}^\xi(t) = \left[A\bar{X}^\xi(t) + \bar{b}(t, \bar{X}^\xi(t)) + \bar{B}(t, \bar{X}_t^\xi) \right] dt + \bar{Q}(t, \bar{X}^\xi(t)) dW(t), \quad \bar{X}_0^\xi = \xi.$$

Define

$$\bar{P}_t f(\xi) = \mathbb{E} f(\bar{X}_t^\xi), \quad t \in [0, T], f \in \mathcal{B}_b(\mathcal{C}).$$

Then it is easy to see that

$$\begin{aligned} P_t f(\xi) &:= \mathbb{E} f(X_t^\xi) = \mathbb{E}(f \circ \theta_t^{-1})(Y_t^\xi) = \mathbb{E}(f \circ \theta_t^{-1})(\bar{X}_t^{\theta_0(\xi)}) \\ &= \bar{P}_t(f \circ \theta_t^{-1})(\theta_0(\xi)), \quad \xi \in \mathcal{C}, t \in [0, T], f \in \mathcal{B}_b(\mathcal{C}). \end{aligned} \quad \boxed{5.12} \quad (5.12)$$

We first study the Harnack inequalities for \bar{P}_t .

To apply the method of coupling by change of measure, we will use the finite dimension approximation argument. More precisely, let $\{\bar{X}^{n, \xi}(t)\}_{t \in [-r, T]}$ solves the finite-dimensional equation on $\mathbb{H}_n := \text{span}\{e_1, \dots, e_n\}$ ($n \geq 1$):

$$\begin{aligned} d\bar{X}^{(n, \xi)}(t) &= \left[A\bar{X}^{(n, \xi)}(t) + \bar{b}^n(t, \bar{X}^{(n, \xi)}(t)) + \bar{B}^n(t, \bar{X}_t^{(n, \xi)}) \right] dt \\ &+ \bar{Q}^n(t, \bar{X}^{(n, \xi)}(t)) dW(t), \quad \bar{X}_0^{(n, \xi)} = \xi \in \mathcal{C}(\mathbb{H}_n), \end{aligned} \quad \boxed{5.13} \quad (5.13)$$

where $\bar{b}^n = \pi_n \bar{b}$, $\bar{B}^n = \pi_n \bar{B}$, $\bar{Q}^n = \pi_n \bar{Q}$. Firstly, we prove that there exists a constant $\tilde{\lambda}(T) \geq \lambda(T)$ such that for any $\lambda \geq \tilde{\lambda}(T)$,

$$\boxed{5.14} \quad (5.14) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left\| \bar{X}_t^\xi - \bar{X}_t^{(n, \pi_n \xi)} \right\|_\infty^2 = 0, \quad t \in [0, T].$$

L5.1 **Lemma 5.1.** Assume **(a1)**, **(a2')** and **(a3'')** with (2.3) in place of (2.5). If in addition $\|B\|_{T,\infty} < \infty$, then there exists a constant $\tilde{\lambda}(T) \geq \lambda(T)$ such that for any $\lambda \geq \tilde{\lambda}(T)$, (5.14) holds.

Proof. For simplicity, we omit ξ and $\pi_n \xi$ from the subscripts, i.e. we write $(\bar{X}_t, \bar{X}_t^{(n)})$ instead of $(\bar{X}_t^\xi, \bar{X}_t^{(n, \pi_n \xi)})$. By Jensen inequality, it suffices to prove there exists a constant $\tilde{\lambda}(T) \geq \lambda(T)$ such that for any $\lambda \geq \tilde{\lambda}(T)$,

$$\textbf{5.15} \quad (5.15) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left\| \bar{X}_t^\xi - \bar{X}_t^{(n, \pi_n \xi)} \right\|_\infty^4 = 0, \quad t \in [0, T].$$

For any $t \in [0, T]$, let

$$\beta_n(t) = \mathbb{E} \sup_{s \in [-r, t]} |\bar{X}(s) - \bar{X}^{(n)}(s)|^4, \quad \beta(t) = \limsup_{n \rightarrow \infty} \beta_n(t).$$

Obviously, **(a1)**, **(a2')** and **(a3'')** imply

$$\mathbb{E} \sup_{t \in [0, T], n \geq 1} (\|\bar{X}_t\|_\infty^4 + \|\bar{X}_t^{(n)}\|_\infty^4) < \infty,$$

so that $\beta(t) < \infty$.

Combining (5.11) with (5.13), it holds that

$$\begin{aligned} \beta_n(t) &\leq 686 \|\xi - \pi_n \xi\|_\infty^4 \\ &\quad + 343 \mathbb{E} \sup_{q \in [0, t]} \left| \int_0^q e^{A(q-s)} [\bar{b}^n(s, \bar{X}(s)) - \bar{b}^n(s, \bar{X}^{(n)}(s))] ds \right|^4 \\ &\quad + 343 \mathbb{E} \sup_{q \in [0, t]} \left| \int_0^q e^{A(q-s)} [\bar{b}(s, \bar{X}(s)) - \bar{b}^n(s, \bar{X}(s))] ds \right|^4 \\ &\quad + 343 \mathbb{E} \sup_{q \in [0, t]} \left| \int_0^q e^{A(q-s)} [\bar{B}^n(s, \bar{X}_s) - \bar{B}^n(s, \bar{X}_s^{(n)})] ds \right|^4 \\ \textbf{5.16} \quad (5.16) \quad &\quad + 343 \mathbb{E} \sup_{q \in [0, t]} \left| \int_0^q e^{A(q-s)} [\bar{B}(s, \bar{X}_s) - \bar{B}^n(s, \bar{X}_s)] ds \right|^4 \\ &\quad + 343 \mathbb{E} \sup_{q \in [0, t]} \left| \int_0^q e^{A(q-s)} [\bar{Q}(s, \bar{X}(s)) - \bar{Q}^n(s, \bar{X}(s))] dW(s) \right|^4 \\ &\quad + 343 \mathbb{E} \sup_{q \in [0, t]} \left| \int_0^q e^{A(q-s)} [\bar{Q}^n(s, \bar{X}(s)) - \bar{Q}^n(s, \bar{X}^{(n)}(s))] dW(s) \right|^4 \\ &=: \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5 + \Gamma_6 + \Gamma_7. \end{aligned}$$

Firstly, for any $\xi \in \mathcal{C}$, $n \geq 1$, by the definition of π_n , we have $\|\xi - \pi_n \xi\|_\infty < \|\xi\|_\infty$ and $|(\xi - \pi_n \xi)(s_1) - (\xi - \pi_n \xi)(s_2)| \leq |\xi(s_1) - \xi(s_2)|$ for any $s_1, s_2 \in [-r, 0]$. Since for any $s \in [-r, 0]$, $|(\xi - \pi_n \xi)(s)| \rightarrow 0$ as $n \rightarrow \infty$, it follows from Arzela-Ascoli Theorem that

$$\textbf{5.17} \quad (5.17) \quad \lim_{n \rightarrow \infty} \Gamma_1 = 0.$$

Similarly to the estimate I_1 in Proposition 3.3, there exists a constant $\tilde{\lambda}(T) \geq \lambda(T)$ such that for any $\lambda \geq \tilde{\lambda}(T)$,

$$\boxed{5.18} \quad (5.18) \quad \Gamma_2 \leq C(\lambda, T) \int_0^t \beta_n(s) ds + \frac{1}{5} \mathbb{E} \beta_n(t).$$

Next, by Hölder inequality and $(\mathbf{a3}'')$, for any $\delta \in (0, 2)$, it holds that

$$\begin{aligned} \Gamma_3 &\leq C e^{\lambda T} \left\{ \int_0^t \left\| (\lambda - A)^{\frac{1}{2}} e^{-(\lambda - A)s} \right\|^\delta ds \right\}^{\frac{4}{\delta}} \\ &\quad \times \mathbb{E} \left\{ \int_0^t \left| (\lambda - A)^{\frac{1}{2}} (u - \pi_n u)(s, \theta^{-1}(s, \bar{X}(s))) \right|^{\frac{\delta}{\delta-1}} ds \right\}^{\frac{4(\delta-1)}{\delta}} \\ &\leq C(\lambda, T, \delta) \mathbb{E} \left\{ \int_0^t \left| (\lambda - A)^{\frac{1}{2}} (u - \pi_n u)(s, \theta^{-1}(s, \bar{X}(s))) \right|^{\frac{\delta}{\delta-1}} ds \right\}^{\frac{4(\delta-1)}{\delta}}. \end{aligned}$$

Combing the definition of π_n and Lemma 3.1 (2) for $a(x) = x^{\frac{1}{2}}$, it follows from dominated convergence theorem that

$$\boxed{5.19} \quad (5.19) \quad \lim_{n \rightarrow \infty} \Gamma_3 = 0.$$

Moreover, $(\mathbf{a2}')$, Hölder inequality and the boundedness of B yield that

$$\boxed{5.20} \quad (5.20) \quad \Gamma_4 \leq C(\lambda, T) \int_0^t \beta_n(s) ds.$$

Again using the boundedness of B , Hölder inequality and dominated convergence theorem, we obtain

$$\boxed{5.21} \quad (5.21) \quad \lim_{n \rightarrow \infty} \Gamma_5 = 0.$$

Furthermore, combining $(\mathbf{a2}')$ with Lemma 1.1, applying dominated convergence theorem, it is easy to see that

$$\boxed{5.22} \quad (5.22) \quad \lim_{n \rightarrow \infty} \Gamma_6 = 0.$$

Finally, combining $(\mathbf{a2}')$ with Lemma 1.1, we have

$$\boxed{5.23} \quad (5.23) \quad \Gamma_7 \leq C(\lambda, T) \int_0^t \beta_n(s) ds.$$

Combining (5.16)-(5.23), applying dominated convergence theorem, it holds that

$$\beta(t) \leq C \int_0^t \beta(s) ds.$$

Since $\beta(t) < \infty$, Gronwall inequality yields $\beta(t) = 0$, $t \in [0, T]$, which implies (5.14). \square

L5.2 **Lemma 5.2.** Assume **(a1)**, **(a2')** with $B = 0$, (2.4) and (2.5). Then for any $\lambda \geq \lambda(T)$, there exists a constant $C(T) > 0$ such that

5.24 (5.24) $\|\nabla u(t, x) - \nabla u(t, y)\|_{\text{HS}} \leq C(T)|x - y|, \quad x, y \in \mathbb{H}, t \in [0, T].$

Proof. In order to prove (5.24), by **(a1)**, it suffices to prove

5.25 (5.25) $\left\| (-A)^{\frac{1-\varepsilon}{2}} [\nabla u(t, x) - \nabla u(t, y)] \right\| \leq C(T)|x - y|, \quad x, y \in \mathbb{H}, t \in [0, T].$

In fact, if (5.25) holds, then

$$\begin{aligned} \|\nabla u(t, x) - \nabla u(t, y)\|_{\text{HS}}^2 &= \left\| (-A)^{\frac{\varepsilon-1}{2}} (-A)^{\frac{1-\varepsilon}{2}} [\nabla u(t, x) - \nabla u(t, y)] \right\|_{\text{HS}}^2 \\ &\leq C(T) \left\| (-A)^{\frac{\varepsilon-1}{2}} \right\|_{\text{HS}}^2 |x - y|^2, \quad x, y \in \mathbb{H}, t \in [0, T]. \end{aligned}$$

Define

$$(R_{s,t}^\lambda f)(x) = \int_s^t e^{-(q-s)\lambda} [P_{s,q}^0 f(q, \cdot)](x), \quad x \in \mathbb{H}, \lambda \geq 0, t \geq s \geq 0, f \in \mathcal{B}_b([0, \infty) \times \mathbb{H}; \mathbb{H}).$$

Firstly, by (2.4) and (5.1), we have

5.26 (5.26) $\|(-A)^{\frac{1}{2}}(\nabla_b u + b)\|_{T,\infty} < \infty$

for any $\lambda \geq \lambda(T)$.

Then it follows from (3.4), (3.5), (5.26), [1, Lemma 2.2 (1)] and dominated convergence theorem that for any $\lambda \geq \lambda(T)$,

5.27 (5.27)
$$\begin{aligned} &\left\| (-A)^{\frac{1}{2}} [\nabla u(t, x) - \nabla u(t, y)] \right\| \\ &= \left\| \nabla \left(R_{t,T}^\lambda \left((-A)^{\frac{1}{2}} (\nabla_b u + b) \right) \right) (x) - \nabla \left(R_{t,T}^\lambda \left((-A)^{\frac{1}{2}} (\nabla_b u + b) \right) \right) (y) \right\| \\ &\leq C|x - y| \log \left(e + \frac{1}{|x - y|} \right) \quad x, y \in \mathbb{H}, t \in [0, T] \end{aligned}$$

holds for some constant $C > 0$. Combining this with (5.26) and (2.5), it is easy to see that for any $\lambda \geq \lambda(T)$,

5.28 (5.28) $\left| (-A)^{\frac{1-\varepsilon}{2}} (\nabla_b u + b)(t, x) - (-A)^{\frac{1-\varepsilon}{2}} (\nabla_b u + b)(t, y) \right| \leq \tilde{\phi}(|x - y|), \quad t \in [0, T], x, y \in \mathbb{H},$

where $\tilde{\phi}(s) = c\sqrt{\phi^2(s) + s}$ with a constant $c > 0$.

Finally, by (3.4), (3.5), (5.28), dominated convergence theorem and [1, Lemma 2.2 (3)], for any $\lambda \geq \lambda(T)$, we conclude that

5.29 (5.29)
$$\begin{aligned} &\left\| (-A)^{\frac{1-\varepsilon}{2}} [\nabla u(t, x) - \nabla u(t, y)] \right\| \\ &= \left\| \nabla \left(R_{t,T}^\lambda \left((-A)^{\frac{1-\varepsilon}{2}} (\nabla_b u + b) \right) \right) (x) - \nabla \left(R_{t,T}^\lambda \left((-A)^{\frac{1-\varepsilon}{2}} (\nabla_b u + b) \right) \right) (y) \right\| \\ &\leq C(T)|x - y|, \quad x, y \in \mathbb{H}, t \in [0, T]. \end{aligned}$$

for a constant $C(T) > 0$. Thus (5.25) holds, and we complete the proof. \square

L5.3 **Lemma 5.3.** Assume **(a1)**, **(a2')**, (2.4) and (2.5). If in addition $\|B\|_{T,\infty} < \infty$ and moreover

$$\boxed{5.30} \quad (5.30) \quad \|Q(t, x) - Q(t, y)\|_{\text{HS}}^2 \leq C(T)|x - y|^2, \quad t \in [0, T], x, y \in \mathbb{H},$$

where $C(T)$ is a positive constant. Then for any $\lambda \geq \lambda(T)$, there exists $K_1 \geq 0$, $K_2 \geq 0$, $K_3 > 0$ and $K_4 \in \mathbb{R}$ (K_1, K_2, K_3, K_4 only depend on T) such that

$$\boxed{5.31} \quad (5.31) \quad |(\bar{Q}^*(\bar{Q}\bar{Q}^*)^{-1})(t, \eta(0))\{\bar{B}(t, \xi) - \bar{B}(t, \eta)\}|_{\mathbb{H}} \leq K_1\|\xi - \eta\|_{\infty};$$

$$\boxed{5.32} \quad (5.32) \quad \|\bar{Q}(t, x) - \bar{Q}(t, y)\| \leq K_2(1 \wedge |x - y|);$$

$$\boxed{5.33} \quad (5.33) \quad \|(\bar{Q}^*(\bar{Q}\bar{Q}^*)^{-1})(t, x)\| \leq K_3;$$

$$\boxed{5.34} \quad (5.34) \quad \|\bar{Q}(t, x) - \bar{Q}(t, y)\|_{HS}^2 + 2\langle x - y, Ax - Ay + \bar{b}(t, x) - \bar{b}(t, y) \rangle \leq K_4|x - y|^2;$$

hold for $t \in [0, T]$, $\xi, \eta \in \mathcal{C}$, and $x, y \in \mathbb{H}$.

Proof. Fix $\lambda \geq \lambda(T)$.

(a) Since $\nabla\theta(t, \cdot) = I + \nabla u(t, \cdot)$, $t \in [0, T]$, (5.1) yields that for any $\lambda \geq \lambda(T)$ and $(t, x) \in [0, T] \times \mathbb{H}$, $\nabla\theta(t, x)$, $(\nabla\theta(t, x))^* \in \mathcal{L}(\mathbb{H}, \mathbb{H})$ are invertible. Then from (5.10),

$$\boxed{5.35} \quad (5.35) \quad (\bar{Q}^*(\bar{Q}\bar{Q}^*)^{-1})(t, x) = (Q^*(QQ^*)^{-1})(t, \theta^{-1}(t, x)) [\nabla\theta(t, \theta^{-1}(t, x))]^{-1}$$

From **(a2')**, (5.33) holds with $K_3 = \frac{8}{7}C_{B,Q}^2(T)$.

(b) Due to (a), in order to prove (5.31), we only need to estimate $|\bar{B}(t, \xi) - \bar{B}(t, \eta)|_{\mathbb{H}}$. From (5.9), **(a2')**, (5.1), (5.2), we have

$$\begin{aligned} & |\bar{B}(t, \xi) - \bar{B}(t, \eta)| \\ &= |\nabla\theta(t, \theta^{-1}(t, \xi(0)))B(t, \theta_t^{-1}(\xi)) - \nabla\theta(t, \theta^{-1}(t, \eta(0)))B(t, \theta_t^{-1}(\eta))| \\ &\leq |\nabla\theta(t, \theta^{-1}(t, \xi(0)))B(t, \theta_t^{-1}(\xi)) - \nabla\theta(t, \theta^{-1}(t, \xi(0)))B(t, \theta_t^{-1}(\eta))| \\ &\quad + |\nabla\theta(t, \theta^{-1}(t, \xi(0)))B(t, \theta_t^{-1}(\eta)) - \nabla\theta(t, \theta^{-1}(t, \eta(0)))B(t, \theta_t^{-1}(\eta))| \\ \boxed{5.36} \quad (5.36) \quad &\leq \sup_{(t,x) \in [0,T] \times \mathbb{H}} \|\nabla\theta(t, x)\| \|\nabla B\|_{T,\infty} \sup_{t \in [0,T], \xi \in \mathcal{C}} \|\nabla\theta_t^{-1}(\xi)\| \|\xi - \eta\|_{\infty} \\ &\quad + \sup_{(t,x) \in [0,T] \times \mathbb{H}} \|\nabla^2\theta(t, x)\| \sup_{(t,x) \in [0,T] \times \mathbb{H}} \|\nabla\theta^{-1}(t, x)\| \|B\|_{T,\infty} |\xi(0) - \eta(0)| \\ &\leq K\|\xi - \eta\|_{\infty}, \quad K > 0. \end{aligned}$$

Combining (5.36) with (5.33), we prove (5.31).

(c) Similarly, from (5.10), again using $(\mathbf{a2}')$, (5.1), (5.2), we arrive at

$$\begin{aligned}
& \|\bar{Q}(t, x) - \bar{Q}(t, y)\| \\
&= \|\nabla\theta(t, \theta^{-1}(t, x))Q(t, \theta^{-1}(t, x)) - \nabla\theta(t, \theta^{-1}(t, y))Q(t, \theta^{-1}(t, y))\| \\
&\leq \|\nabla\theta(t, \theta^{-1}(t, x))Q(t, \theta^{-1}(t, x)) - \nabla\theta(t, \theta^{-1}(t, x))Q(t, \theta^{-1}(t, y))\| \\
&+ \|\nabla\theta(t, \theta^{-1}(t, x))Q(t, \theta^{-1}(t, y)) - \nabla\theta(t, \theta^{-1}(t, y))Q(t, \theta^{-1}(t, y))\| \\
\boxed{5.37} \quad (5.37) \quad &\leq \sup_{(t,x) \in [0,T] \times \mathbb{H}} \|\nabla\theta(t, x)\| \|\nabla Q\|_{T,\infty} \sup_{(t,x) \in [0,T] \times \mathbb{H}} \|\nabla\theta^{-1}(t, x)\| |x - y| \\
&+ \sup_{(t,x) \in [0,T] \times \mathbb{H}} \|\nabla^2\theta(t, x)\| \|Q\|_{T,\infty} \sup_{(t,x) \in [0,T] \times \mathbb{H}} \|\nabla\theta^{-1}(t, x)\| |x - y| \\
&\leq K' |x - y|, \quad K' > 0,
\end{aligned}$$

and

$$\boxed{5.38} \quad (5.38) \quad \|\bar{Q}(t, x) - \bar{Q}(t, y)\| \leq 2 \sup_{(t,x) \in [0,T] \times \mathbb{H}} \|\nabla\theta(t, x)\| \|Q\|_{T,\infty} \leq K'', \quad K'' > 0.$$

Then (5.37) and (5.38) yield (5.32).

(d) Finally, from (5.10), applying $(\mathbf{a2}')$, (5.1), (5.2), (5.30) and Lemma 5.2, we obtain

$$\begin{aligned}
& \|\bar{Q}(t, x) - \bar{Q}(t, y)\|_{\text{HS}} \\
&= \|\nabla\theta(t, \theta^{-1}(t, x))Q(t, \theta^{-1}(t, x)) - \nabla\theta(t, \theta^{-1}(t, y))Q(t, \theta^{-1}(t, y))\|_{\text{HS}} \\
&\leq \|\nabla\theta(t, \theta^{-1}(t, x))Q(t, \theta^{-1}(t, x)) - \nabla\theta(t, \theta^{-1}(t, x))Q(t, \theta^{-1}(t, y))\|_{\text{HS}} \\
&+ \|\nabla\theta(t, \theta^{-1}(t, x))Q(t, \theta^{-1}(t, y)) - \nabla\theta(t, \theta^{-1}(t, y))Q(t, \theta^{-1}(t, y))\|_{\text{HS}} \\
\boxed{5.39} \quad (5.39) \quad &\leq \sup_{(t,x) \in [0,T] \times \mathbb{H}} \|\nabla\theta(t, x)\| \|Q(t, \theta^{-1}(t, x)) - Q(t, \theta^{-1}(t, y))\|_{\text{HS}} \\
&+ \|Q\|_{T,\infty} \|\nabla u(t, \theta^{-1}(t, x)) - \nabla u(t, \theta^{-1}(t, y))\|_{\text{HS}} \\
&\leq C(T) \left[\sup_{t \in [0,T] \times \mathbb{H}} \|\nabla\theta(t, x)\| + \|Q\|_{T,\infty} \right] \sup_{t \in [0,T] \times \mathbb{H}} \|\nabla\theta^{-1}(t, x)\| |x - y| \\
&\leq K_0 |x - y|, \quad K_0 > 0.
\end{aligned}$$

Moreover, (5.8) (5.1), (5.2), and $(\mathbf{a3}'')$ yield that

$$\begin{aligned}
& \langle A(x - y), x - y \rangle + \langle (-A)[u(t, \theta^{-1}(t, x)) - u(t, \theta^{-1}(t, y))], x - y \rangle \\
\boxed{5.40} \quad (5.40) \quad &= - \left| (-A)^{\frac{1}{2}}(x - y) \right|^2 + \left\langle (-A)^{\frac{1}{2}} [u(t, \theta^{-1}(t, x)) - u(t, \theta^{-1}(t, y))] , (-A)^{\frac{1}{2}}(x - y) \right\rangle \\
&\leq - |(-A)^{\frac{1}{2}}(x - y)|^2 + c|x - y|^2 + \frac{1}{2} \left| (-A)^{\frac{1}{2}}(x - y) \right|^2 \\
&\leq c|x - y|^2
\end{aligned}$$

for a constant $c > 0$. Combining (5.39) with (5.40), we obtain (5.34). \square

Proof of Theorem 2.2. Combining Lemma 5.1 with Lemma 5.3, for any $\lambda \geq \tilde{\lambda}(T)$, using [10, Theorem 3.4.1, Theorem 4.3.1 and Theorem 4.3.2], we obtain the log-Harnack inequality and Harnack inequality with power for \bar{P}_T . Next, we only show Theorem 2.2 (2), and (1) is completely similar.

For every $p > (1 + K_2 K_3)^2$, the Harnack inequality with power

$$\boxed{5.41} \quad (5.41) \quad \bar{P}_T f(\eta) \leq (\bar{P}_T f^p(\xi))^{\frac{1}{p}} \exp \tilde{\Phi}_p(T; \xi, \eta), \quad \xi, \eta \in \mathcal{C}$$

holds for non-negative function $f \in \mathcal{B}_b(\mathcal{C})$, where

$$\tilde{\Phi}_p(T; \xi, \eta) = \tilde{C}(p) \left\{ 1 + \frac{|\xi(0) - \eta(0)|^2}{T - r} + \|\xi - \eta\|_\infty^2 \right\}.$$

and $\tilde{C} : ((1 + K_2 K_3)^2, \infty) \rightarrow (0, \infty)$ is a decreasing function.

From (5.12), we obtain that for every $p > (1 + K_2 K_3)^2$ and non-negative function $f \in \mathcal{B}_b(\mathcal{C})$,

$$\begin{aligned} P_T f(\eta) &= \bar{P}_T(f \circ \theta_T^{-1})(\theta_0(\eta)) \\ \boxed{5.42} \quad (5.42) \quad &\leq \{\bar{P}_T(f^p \circ \theta_T^{-1})(\theta_0(\xi))\}^{\frac{1}{p}} \exp \tilde{\Phi}_p(T; \theta_0(\xi), \theta_0(\eta)) \\ &= \{P_T f^p(\xi)\}^{\frac{1}{p}} \exp \tilde{\Phi}_p(T; \theta_0(\xi), \theta_0(\eta)) \quad \xi, \eta \in \mathcal{C} \end{aligned}$$

Combining (5.2), we have

$$\tilde{\Phi}_p(T; \theta_0(\xi), \theta_0(\eta)) \leq \frac{81}{64} \tilde{C}(p) \left\{ 1 + \frac{|\xi(0) - \eta(0)|^2}{T - r} + \|\xi - \eta\|_\infty^2 \right\} =: \Psi_p(T; \xi, \eta).$$

Letting $K = K_1 K_2$, $C = \frac{81}{64} \tilde{C}$, (5.42) yields (2.8). Thus, we finish the proof. \square

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